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ON THE JOINT DISTRIBUTION OF THE SURPLUS IMMEDIATELY PRIOR TO RUIN AND THE DEFICIT AT RUIN

Veli Bayramov[1*], **Rovshan Aliyev**[1,2]

[1]Baku State University, Baku, Azerbaijan
[2]Institute of Control Systems of ANAS, Baku, Azerbaijan

Abstract. In this paper Erlang(n) risk model with constant interest force is considered. The recursive algorithm is given for the joint distribution of the surplus immediately prior to ruin and the deficit at ruin.

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Corresponding author: Veli Bayramov, Baku State University, Z.Khalilov st. 23, AZ1148, Baku, Azerbaijan, e-mail: *veli_bayramov@yahoo.com*
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1 Introduction

In the risk theory, there is considerable interest in the study of the expected discounted penalty function. Gerber & Shiu (1998) defined the expected discounted penalty function in the classical Poisson risk model, and they derived the defective renewal equation satisfied by the expected value. Lin et al. (2003) investigated the penalty function of the classical Poisson risk model in the presence of a constant dividend barrier, and obtained an integro-differential equation for the expected value. Their works are concentrated on the classical Poisson risk model in which the claim inter-arrival times have exponential distributions. The Erlang(2) no-interest risk model, which assumes that the waiting times between claims are Erlang(2) distributions, generalizes the classical Poisson risk model and receives a remarkable attention. With this model, Dickson & Hipp (2001) and Dickson & Hipp (1998) gave the expressions for the survival probability and the moments of the time to ruin. Xing & Wu (2006) established an integro-differential equation of the expected discounted penalty function of Erlang(n) no-interest risk model. As we know, a large portion of the surplus of the insurance company comes from investment income, and interest factor must affect the management of the company. So it is necessary to discuss the ruin problem by taking into account interest factor. Gaoqin et al. (2006) gave the recursive algorithm for the joint distribution of the surplus immediately prior to ruin and the deficit at ruin of Erlang(2) risk model with constant interest.

In this paper we consider Erlang(n) risk model with a constant interest rate. The recursive algorithm for the joint distribution of the surplus immediately prior to ruin and the deficit at ruin is given.

2 The model

Consider the insurance risk process

$$U_\delta(t) = ue^{\delta t} + c\int_0^t e^{\delta(t-y)}dy - \sum_{i=1}^{N(t)} X_i e^{\delta\left(t-\sum_{j=1}^{i} T_j\right)}, \tag{1}$$

where $u = U_\delta(0) \geq 0$ is the initial capital of the insurance company, $c > 0$ is the premium rate, δ is the constant interest force, $\{X_i, i \geq 1\}$ denotes the sequence of independent and identically distributed (i.i.d.) non-negative successive claims, and $N(t)$ $(t \geq 0)$ denotes the number of claims up to time t, which is a counting process independent of $\{X_i, i \geq 1\}$. If $N(t)$ is a renewal process, that is, the times T_i, $i \geq 1$, elapsed between successive claims are i.i.d., the model above is the renewal risk model, which introduced by Sparre Anderson (1957). In this paper, we consider the Erlang(n) risk model with a constant interest rate, where T_1 has an Erlang(n) probability density function (p.d.f.) $\gamma_n(t)$ with scale parameter $\beta > 0$:

$$\gamma_n(t) = \frac{\beta^n}{(n-1)!} t^{n-1} e^{-\beta t}, \; t > 0. \tag{2}$$

Now define the time of ruin by $T_\delta = \inf\{t : U_\delta(t) < 0\}$, and $T_\delta = \infty$ if $U_\delta(t) \geq 0$ for all $t > 0$. Then the ruin probability is defined as $\Psi_{n;\delta}(u) = \mathrm{P}\{T_\delta < \infty \,|U_\delta(0) = u\}$. Let $U_\delta\left(T_\delta^-\right)$ and $|U_\delta(T_\delta)|$ denote the surplus immediately prior to ruin and deficit at ruin when ruin occurs, respectively. We consider the expected discounted penalty function of the surplus immediately prior to ruin and the deficit at ruin when ruin occurs as a function of the initial surplus u, namely

$$\Phi_{n;\delta,\alpha}(u) = \mathrm{E}\left\{e^{-\alpha T_\delta}\omega\left(U_\delta\left(T_\delta^-\right), |U_\delta(T_\delta)|\right)\mathrm{I}_{(T_\delta<\infty)} \,|U_\delta(0) = u\right\}, \tag{3}$$

where $\alpha \geq 0$, I_B is the indicator function of a set B, and $\omega(x_1, x_2)$, $0 < x_1, x_2 < \infty$, is a non-negative function. Many properties of the surplus process can be obtained from this general function. In the case of $\alpha = 0$, choosing different forms of the function $\omega(x_1, x_2)$, we obtain different information relating to the deficit at ruin and the surplus prior to ruin. For example, $\Phi_{n;\delta,\alpha}(u)$ will represent the ν-order moment of the deficit at ruin (or the surplus prior to ruin) if we specially choose $\omega(x_1, x_2) = x_1^\nu$ (or x_2^ν), represent their joint c.d.f. if $\omega(x_1, x_2) = \mathrm{I}_{(x_1 \leq x, x_2 \leq y)}$, represent c.d.f. of the deficit at ruin if $\omega(x_1, x_2) = \mathrm{I}_{(x_2 \leq x)}$ and so on. When $\omega(x_1, x_2) \equiv 1$, $\Phi_{n;\delta,\alpha}(u)$ reduces to $\Psi_{n;\delta}(u)$. The financial explanations on $\omega(x_1, x_2)$ can be found in Gerber & Shiu (1998).

Throughout this paper we suppose that the claim sizes, X_i, $i \geq 1$, are i.i.d. with a common c.d.f. F supported on $[0, \infty)$, a finite mean μ, and p.d.f. $f(x)$. It is always assumed that the safety loading condition

$$\rho = \frac{c\mathrm{E}\{T_1\} - \mu}{\mu} = \frac{cn/\beta - \mu}{\mu} > 0 \tag{4}$$

holds.

3 Main result

In the following, we will discuss the case where $\alpha = 0$ and $\omega(x_1, x_2) = \mathrm{I}_{(x_1 \leq x, x_2 \leq z)}$, $x > 0$, $z > 0$. Then

$$\Phi_{n;\delta,\alpha}(u) = \mathrm{P}\left\{U_\delta\left(T_\delta^-\right) \leq x, |U_\delta(T_\delta)| \leq z, T_\delta < \infty \,|U_\delta(0) = u\right\},$$

which is the joint distribution of the surplus immediately prior to ruin and the deficit at ruin under interest force. Denote the distribution by $L_n(u, x, z)$, we will give the recursive algorithm of $L_n(u, x, z)$ by the similar method in Gaoqin et al. (2006) and Lin & Wang (2005).

Theorem 1. *Consider the Erlang(n) risk model with the relative safety loading condition under constant interest force, then the joint distribution of the surplus immediately before ruin and the deficit at ruin is*

$$L_n(u, x, z) = \sum_{k=1}^{\infty} l_{n,k}(u, x, z),$$

where

$$l_{n,1}(u,x,z)=\frac{\beta^n}{(n-1)!}\int_0^{\left[\frac{1}{\delta}\ln\left(\frac{x\delta+c}{u\delta+c}\right)\right]_+}\left[F\left(z+ue^{\delta t}+ch_t\right)-F\left(ue^{\delta t}+ch_t\right)\right]t^{n-1}e^{-\beta t}dt,$$

$$l_{n,k}(u,x,z)=\frac{\beta^n}{(n-1)!}\int_0^{\infty}\int_0^{ue^{\delta t}+ch_t}l_{n,k-1}\left(ue^{\delta t}+ch_t-w,x,z\right)f(w)\,t^{n-1}e^{-\beta t}dwdt,\ k\geq 2,$$

$$h_t=\frac{e^{\delta t}-1}{\delta}.$$

Proof. Let $S_k=\sum_{i=1}^k T_i$ and $h_t=\frac{e^{\delta t}-1}{\delta}$, from Lin & Wang (2005) we get

$$U_\delta(t)=ue^{\delta t}+ch_t-\sum_{i=1}^{N(t)}X_ie^{\delta(t-S_i)},U_\delta(S_k)=ue^{\delta\sum_{j=1}^k T_j}-\sum_{i=1}^{k}W_ie^{\delta\sum_{j=i+1}^k T_j},$$

where $W_i=X_i-ch_{T_i}$, $i\geq 1$, and the common distribution of (W_i,T_i) $(i\geq 1)$ is

$$G(w,t)=\frac{\beta^n}{(n-1)!}\int_0^t F(w+ch_s)\,s^{n-1}e^{-\beta s}ds.$$

Note that the ruin can only happen at time period S_i $(i\geq 1)$, we have

$$L_n(u,x,z)=\sum_{k=1}^{\infty}\mathrm{P}^u\left\{U_\delta(S_1)\geq 0,...,U_\delta(S_{k-1})\geq 0,0\leq U_\delta\left(S_k^-\right)\leq x,-z\leq U_\delta(S_k)<0\right\}=$$

$$=\sum_{k=1}^{\infty}l_{n,k}(u,x,z),$$

in which P^u denotes the conditional probability given $U_\delta(0)=u$, and

$$l_{n,k}(u,x,z)=\mathrm{P}^u\left\{U_\delta(S_1)\geq 0,...,U_\delta(S_{k-1})\geq 0,0\leq U_\delta\left(S_k^-\right)\leq x,-z\leq U_\delta(S_k)<0\right\}.$$

By definition we can obtain

$$l_{n,1}(u,x,z)=\mathrm{P}^u\left\{0\leq U_\delta\left(S_1^-\right)\leq x,-z\leq U_\delta(S_1)<0\right\}=$$

$$=\mathrm{P}\left\{ue^{\delta T_1}+ch_{T_1}\leq x,-z\leq ue^{\delta T_1}-W_1<0\right\}=$$

$$=\int_0^{\infty}\int_0^{\infty}\mathrm{P}\left\{ue^{\delta T_1}+ch_{T_1}\leq x,-z\leq ue^{\delta T_1}-W_1<0\middle|W_1=w,T_1=t\right\}dG(w,t)=$$

$$=\frac{\beta^n}{(n-1)!}\int_0^{\left[\frac{1}{\delta}\ln\left(\frac{x\delta+c}{u\delta+c}\right)\right]_+}\left[F\left(z+ue^{\delta t}+ch_t\right)-F\left(ue^{\delta t}+ch_t\right)\right]t^{n-1}e^{-\beta t}dt,$$

where $[m]_+=\max(0,m)$.

$$l_{n,2}(u,x,z)=\mathrm{P}^u\left\{U_\delta(S_1)\geq 0,0\leq U_\delta\left(S_2^-\right)\leq x,-z\leq U_\delta(S_2)<0\right\}=$$

$$=\mathrm{P}\left\{ue^{\delta T_1}-W_1\geq 0,ue^{\delta(T_1+T_2)}-W_1e^{\delta T_2}+ch_{T_2}\leq x,-z\leq ue^{\delta(T_1+T_2)}-W_1e^{\delta T_2}-W_2<0\right\}=$$

$$=\int_0^{\infty}\int_{-ch_t}^{ue^{\delta t}}\mathrm{P}\left\{\left(ue^{\delta t}-w\right)e^{\delta T_2}+ch_{T_2}\leq x,-z\leq\left(ue^{\delta t}-w\right)e^{\delta T_2}-W_2<0\right\}dG(w,t)=$$

$$=\frac{\beta^n}{(n-1)!}\int_0^{\infty}\int_0^{ue^{\delta t}+ch_t}l_{n,1}\left(ue^{\delta t}+ch_t-w,x,z\right)f(w)\,t^{n-1}e^{-\beta t}dwdt.$$

By inductive reasoning, when $k \geq 2$,

$$l_{n,k}(u,x,z) = \mathrm{P}^u \left\{ U_\delta(S_1) \geq 0, ..., U_\delta(S_{k-1}) \geq 0, 0 \leq U_\delta\left(S_k^-\right) \leq x, -z \leq U_\delta(S_k) < 0 \right\} =$$

$$= \mathrm{P}\left\{ ue^{\delta T_1} - W_1 \geq 0, ..., ue^{\delta \sum_{j=1}^{k-1} T_j} - \sum_{i=1}^{k-1} W_i e^{\delta \sum_{j=i+1}^{k-1} T_j} \geq 0, \right.$$

$$\left. ue^{\delta \sum_{j=1}^{k} T_j} - \sum_{i=1}^{k-1} W_i e^{\delta \sum_{j=i+1}^{k} T_j} + ch_{T_k} \leq x, -z \leq ue^{\delta \sum_{j=1}^{k} T_j} - \sum_{i=1}^{k} W_i e^{\delta \sum_{j=i+1}^{k} T_j} < 0 \right\} =$$

$$= \int_0^\infty \int_{-ch_t}^{ue^{\delta t}} \mathrm{P}\left\{ \left(ue^{\delta t} - w\right) e^{\delta T_2} - W_2 \geq 0, ..., \left(ue^{\delta t} - w\right) e^{\delta \sum_{j=2}^{k-1} T_j} - \sum_{i=2}^{k-1} W_i e^{\delta \sum_{j=i+1}^{k-1} T_j} \geq 0, \right.$$

$$\left(ue^{\delta t} - w\right) e^{\delta \sum_{j=2}^{k} T_j} - \sum_{i=2}^{k-1} W_i e^{\delta \sum_{j=i+1}^{k} T_j} + ch_{T_k} \leq x,$$

$$\left. -z \leq \left(ue^{\delta t} - w\right) e^{\delta \sum_{j=2}^{k} T_j} - \sum_{i=2}^{k} W_i e^{\delta \sum_{j=i+1}^{k} T_j} < 0 \right\} dG(w,t) =$$

$$= \frac{\beta^n}{(n-1)!} \int_0^\infty \int_0^{ue^{\delta t} + ch_t} l_{n,k-1}\left(ue^{\delta t} + ch_t - w, x, z\right) f(w)\, t^{n-1} e^{-\beta t} dw dt.$$

$\square$

This completes the proof.

If we take $n = 2$ in Theorem 1, we can obtain Theorem 3 in Gaoqin et al. (2006).

4 Conclusion

The function (3) is a general function, which plays an important role for investigating risk insurance processes, especially, when there is an investment income, because many properties of the surplus process can be obtained from this general function as described in Section 2. So, considering any special case of this general function helps us to investigate risk insurance processes.

In this paper Erlang(n) risk model with constant interest force is considered. The recursive algorithm for the joint distribution of the surplus immediately prior to ruin and the deficit at ruin is given by the similar method in Gaoqin et al. (2006) and Lin & Wang (2005). There also shown that how we can obtain other results in literature.

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References

Anderson E.S. (1957). On the collective theory of risk in the class of contagious between the claims. In *Transactions of the XV International Congress of Actuaries.*

Dickson D.C.M. & Hipp C. (2001). On the time to ruin for Erlang(2) risk processes. *Insurance: Mathematics and Economics*, *29*(3), 333-344.

Dickson D.C.M. & Hipp C. (1998). Ruin probabilities for Erlang(2) risk processes. *Insurance: Mathematics and Economics*, *22*(3), 251-262.

Gaoqin N., Cihua L. & Lixia X. (2006). Expected discounted penalty function of Erlang(2) risk model with constant interest. *Appl. Math. J. Chinese Univ. Ser. B*, *21*(3), 243-251.

Gerber H.U. & Shiu E.S.W. (1998). On the time value of ruin. *North American Actuarial Journal*, *2*(1), 48-78.

Lin Q.M. & Wang R.M. (2005). Calculation of ruin probabilities under a renewal risk model with interest force. *Journal of East China Normal University (Natural Science)*, 1, 46-52 (in Chinese).

Lin X.S., Willmot G.E. & Drekic S. (2003). The classical risk model with a constant dividend barrier: analysis of the Gerber-Shiu discounted penalty function. *Insurance: Mathematics and Economics, 33*(3), 551-556.

Xing Y. & Wu R. (2006). Moments of the time of ruin, surplus before ruin and the deficit at ruin in the Erlang(n) risk process. *Acta Mathematicae Applicatae Sinica*, English Series, *22*(4), 599-606.